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SOLUTION OF DIFFERENTIAL EQUATIONS FOR CONICAL SHELLS WITH A SINGULAR RIGHT-SIDE PART

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The problem of determining the stress-strain state of shells under localized and concentrated force and temperature actions can be reduced to differential equations with a singular right-hand side. In this case, as for differential equations describing the behavior of cylindrical shells, application of asymptotic synthesis methods (ASM) is effective. These methods are based on "sewing" solutions of lower-order equations of a simpler structure [1, 2].

If one accepts the classical Kirchhoff–Love hypothesis, for conical circular shells, the governing equation can be written as [3, 4]:

$$x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}+1\right)\left[x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}+1\right)\tilde{N}+\frac{2}{\sin^2\theta}\frac{\partial^2\tilde{N}}{\partial\beta^2}-\frac{ic_0\cot\theta}{h}x\tilde{N}\right]+\frac{1}{\sin^4\theta}\left(\frac{\partial^4\tilde{N}}{\partial\beta^4}+\frac{\partial^2\tilde{N}}{\partial\beta^2}\right)$$
$$=-\frac{ic_0}{h}\left[x\frac{\partial}{\partial x}\left(x\frac{\partial}{\partial x}+1\right)x^2p+\left(x\frac{\partial}{\partial x}+1\right)x^2p+\frac{1}{\sin^2\theta}\frac{\partial^2x^2p}{\partial\beta^2}\right]. \quad (1)$$

Here, x and β are the coordinates of an arbitrary point on the shell surface; h is the shell thickness; 2θ is the tapering angle; $c_0=\sqrt{12(1-\nu^2)}$; ν is Poisson's coefficient; $\tilde{N}(x,\beta)=N_x(x,\beta)+N_\beta(x,\beta)$ is a governing complex force function.

Let k forces that are uniformly distributed along the boundary act at the cross section $x=x_0$, then the loading is

$$p(x,\beta)=\frac{P}{AB}\delta(x-x_0,\beta-\beta_0),\quad A=1,\quad B=x\sin\theta, \quad (2)$$

where $\delta(x-x_0,\beta-\beta_0)$ is a Dirac δ-function; P is the concentrated force.

After substituting into (1) the δ-function and also the function $\tilde{N}(x,\beta)$, which are represented as series in terms of β,

$$\delta(x-x_0,\beta-\beta_0)=\frac{1}{\pi}\delta(x-x_0)\left[\frac{1}{2}k+\sum_{n=1}^{\infty}\cos kn(\beta-\beta_0)\right],\quad \tilde{N}(x,\beta)=\sum_{n=0}^{\infty}\tilde{N}_n(x)\cos kn(\beta-\beta_0),$$

and changing the variable $z=(ic_0\cot\theta)x/h$, one passes to the ordinary differential equation

$$\left\{z\frac{d}{dz}\left(z\frac{d}{dz}+1\right)\left[z\frac{d}{dz}\left(z\frac{d}{dz}+1\right)-z-\frac{2k^2n^2}{\sin^2\theta}\right]+\frac{k^2n^2(k^2n^2-1)}{\sin^4\theta}\right\}\tilde{N}_n$$
$$=-\frac{ic_0}{\pi h}\frac{P}{\sin\theta}\left[z\frac{d}{dz}\left(z\frac{d}{dz}+1\right)z\delta(z-z_0)+\left(z\frac{d}{dz}+1\right)z\delta(z-z_0)-\frac{k^2n^2}{\sin^2\theta}z\delta(z-z_0)\right]. \quad (3)$$

The solution of Eq. (3) with zero right-hand side is determined using generalized hypergeometric functions. The partial integral of the nonhomogeneous equation can be determined by the method of variation

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of constants. The resulting integrals can be computed provided the right-hand side contains the Dirac δ-function, by using its filtering property

$$f(z)\delta(z-z_0)=f(z_0)\delta(z-z_0),$$

from which we have

$$f(z)\delta'(z-z_0)=f(z_0)\delta'(z-z_0)-f'(z)\Big|_{z=z_0}\delta(z-z_0),\qquad \int f(z)\delta(z-z_0)dz=f(z_0)H(z-z_0)$$

[$H(z-z_0)=1$ ($z>z_0$), 0 ($z<z_0$) is a unit Heaviside function].

The solution of the homogeneous equation (3) can be written in terms of the hypergeometric function as (at $n\geqslant 2$)

$$\tilde{N}_n=\sum_{l=1}^{4}\tilde{C}_l z^{\rho_l}{}_2F_3(1+\rho_l,2+\rho_l;1+\rho_l-\rho_1,*1+\rho_l-\rho_2,1+\rho_l-\rho_3,1+\rho_l-\rho_4;z),$$

where

$$\rho_{1-4}=\frac{1}{2}\left[-1\pm\sqrt{1+4kn(kn\pm 1)\sin^{-2}\theta}\right]$$

are roots of the determining equation; the asterisk indicates that the term with $l=j$ must be excluded from terms of the form $1+\rho_l-\rho_j$.

The general solution of Eq. (3) for $n>2$ takes the form [3, 4]

$$\tilde{N}_n=\frac{ic_0}{\pi h}\frac{P}{\sin\theta}\sum_{l=1}^{4}y_l(z)\Bigg\{\tilde{C}_l+(-1)^{l-1}\Bigg\{H(z-z_0)\left[\left(4-\frac{k^2n^2}{\sin^2\theta}\right)f_{1l}(z)-5f'_{2l}(z)+f''_{3l}(z)\right]_{z=z_0}$$
$$+\delta(z-z_0)\Big[5f_{2l}(z)-2f'_{3l}(z)\Big]_{z=z_0}+f_{3l}(z_0)\delta'(z-z_0)\Bigg\}\Bigg\}.$$

Here,

$$C_l=(-1)^{l-1}\frac{ic_0}{\pi h}\frac{P}{\sin\theta}\Bigg\{H(z-z_0)\left[\left(4-\frac{k^2n^2}{\sin^2\theta}\right)f_{1l}(z)-5f'_{2l}(z)+f''_{3l}(z)\right]_{z=z_0}$$
$$+\delta(z-z_0)\Big[5f_{2l}(z)-2f'_{3l}(z)\Big]_{z=z_0}+f_{3l}(z_0)\delta'(z-z_0)\Bigg\};$$

$$f_{1l}=\frac{z\varphi_l}{W_n};\quad f_{2l}(z)=\frac{z^2\varphi_l}{W_n};\quad f_{3l}(z)=\frac{z^3\varphi_l}{W_n};$$

φ_l are minors; $W_n(y_1(z),y_2(z),y_3(z),y_4(z))$ is the Wronskian, where

$$y_l(z)=z^{\rho_l}{}_2F_3(1+\rho_l,2+\rho_l;1+\rho_l-\rho_1\ldots*1+\rho_l-\rho_4;z);$$

$$y_l^{(m)}(z)=\rho_l(\rho_l-1)\ldots(\rho_l-m+1)z^{\rho_l-m}{}_3F_4(1+\rho_l,2+\rho_l,1+\rho_l;1+\rho_l-\rho_1\ldots1+\rho_l-\rho_4;z).$$

Numerical realization of this type of solutions with allowance for relationships between the solving complex effort and other unknown factors involves certain difficulties even in the case of a concentrated load discussed in this paper. Calculation procedures become complicated when one uses the obtained solutions as a Green's function. Therefore, to obtain an effective solution of the formulated problem, one should apply asymptotic synthesis methods according to which the solution of the governing equation (1) is based upon approximate equations of the edge-effect type, semimomentless theory, equations of shallow shells, and of flexural and tangential states. Solutions of the approximate equations are "sewed" for numbers $\bar{n}$ and n^* determined by the formulas [1, 2]

$$k^4n^4\approx 2\sqrt{3}(R_0/h),\quad k^4n^4\approx 2(1-\nu^2)(R_0/h)^{5/2}$$

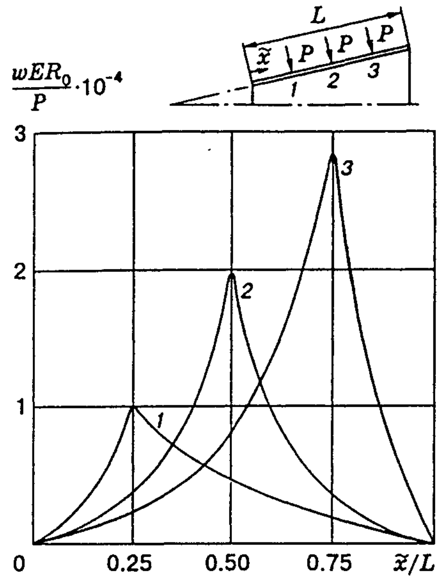

Fig. 1

respectively (one takes the integer values of n from these formulas), where R_0 is the shell radius corresponding to the application point of the concentrated load. When a load is applied to rectangular, elliptic, or circular regions, it corresponds to the radius circle of the going through the center of the region.

In the case of semimomentless theory, which describes the so-called basic stress-strain state of a conical shell, the solving equation with respect to the amplitude value $v_n(x)$ of the circumferential displacement has the form [5]

$$\frac{d^2}{dx^2}\left\{x\left[\tan^2\theta + c_0^2\frac{x^2}{h^2}\right]\frac{d^2}{dx^2}v_n(x)\right\} - 2\mu_n^2\tan\theta\left\{\frac{1}{x}\frac{d^2}{dx^2}v_n(x) + \frac{d^2}{dx^2}\left[\frac{1}{x}v_n(x)\right]\right\} + 4\frac{\mu_n^4}{x^3}v_n(x)$$

$$= \frac{2knP}{\pi D\sin 2\theta}\left\{\frac{1}{x}\frac{d}{dx}\left[x^2\delta(x-x_0)\right] + \left(1 - \frac{k^2n^2}{\sin^2\theta}\right)\delta(x-x_0)\right\}, \tag{4}$$

where D is the cylindrical rigidity.

The edge effect immediately adjacent to the force application zone (which is concentrated for a concentrated force and local for loading of a finite region) is described by the equation for the amplitude value of the normal displacement [5]

$$\frac{d^2}{dx^2}\left[x\frac{d^2}{dx^2}w_n(x)\right] - \frac{d}{dx}\left[\frac{1}{x}\frac{d}{dx}w_n(x)\right] + \frac{c_0^2}{xh^2\tan^2\theta}w_n(x) = \frac{P}{\pi D\sin\theta}\delta(x-x_0). \tag{5}$$

The stress state with a high variability is accurately determined from the equations of shallow shell theory

$$\frac{1}{Eh}\nabla^2\nabla^2\varphi - \frac{\theta}{x}\frac{\partial^2 w}{\partial x^2} = 0, \qquad \frac{\theta}{x}\frac{\partial^2\varphi}{\partial x^2} + D\nabla^2\nabla^2 w = p(x,\beta) \qquad \left(\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{1}{x}\frac{\partial}{\partial x} + \frac{1}{x^2}\frac{\partial^2}{\partial\beta^2}\right).$$

These equations are reduced to an equation for the function $\tilde{F}$

$$\nabla^2\nabla^2\tilde{F} - \frac{i\theta c_0}{h}\frac{1}{x}\frac{\partial^2\tilde{F}}{\partial x^2} = \frac{1}{D}p(x,\beta), \quad \tilde{F} = w + \frac{i\theta c_0}{Eh^2}\varphi. \tag{6}$$

If we write the solving function $\tilde{F}(x,\beta)$ in the form

$$\tilde{F}(x,\beta) = \sum_{n=0}^{\infty} \tilde{F}_n(x) \cos kn(\beta - \beta_0) \tag{7}$$

and substitute (2) and (7) into (6), making the replacement $z = a_0 x$, $a_0 = i\theta c_0 h^{-1}$, we get

$$\nabla_n^2 \nabla_n^2 \tilde{F}_n(z) - \frac{1}{z}\frac{d^2}{dz^2}\tilde{F}_n(z) = \frac{P}{\pi D a_0^2}\frac{1}{z}\delta(z - z_0). \tag{8}$$

At large harmonic numbers ($n > n^*$), the system of equations is split into two independent equations, one of which describes the flexural state and the other, the tangential state.

In the case of the flexural state, one uses instead of Eq. (6) the following simpler equation with respect to the amplitude value of the normal displacement $w_n(x)$:

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{k^2n^2}{x^2}\right)^2 w_n(x) = \frac{P}{\pi D}\delta(x - x_0). \tag{9}$$

After determination of the amplitude values of the solving functions using Eqs. (3)–(5), (8), and (9), and the amplitude values of the sought-for displacements, forces, and moments, the further procedure of deriving the solution is reduced to "sewing," at harmonic numbers $\bar{n}$ and n^*, various elementary solutions, depending on the specific AMS used [1, 2].

We present the values of the normal displacement at the application point ($\tilde{x}_0/R_0 = 2,6$) of the concentrated force P for a conical shell with freely supported edges and with parameters $L/R_0 = 5.2$, $R_0/h = 100$, and $\theta = 11°$:

Theory (method)	1	2	3	4	5	6
wER_0P^{-1}	19371	19147	21938	19355	19655	19707

Here, numbers 1–6 correspond to the general theory of shells, shallow shell theory, semimomentless theory with an edge effect, and the 1st, 2nd, and 3rd AMS.

Curves 1–3 in Fig. 1 show the change in the radial displacement along the zero ruling of the shell ($\beta = 0$) for the case of application of the concentrated force P at one of three points located at distances $\tilde{x} = 0.25L$, $0.5L$, and $0.75L$ from the edge of the shell with a smaller diameter. Curves 1–3 correspond to the cases where forces are applied at points 1–3. The numerical information was obtained on the basis of the second AMS.

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